TOPOLOGICAL PRESSURE FOR SUB-ADDITIVE POTENTIALS OF AMENABLE GROUP ACTIONS

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ABSTRACT. The topological pressure for any sub-additive potentials of a countable discrete amenable group action and any given open cover is defined. A local variational principle for the topological pressure is established.

1. Introduction and main result

Entropies are fundamental to our current understanding of dynamical systems. The classical measure-theoretic entropy for an invariant measure and the topological entropy were introduced in [21] and [1] respectively, and the classical variational principle was completed in [11, 12]. Since then a subject involving to define new measure-theoretic and topological notations of entropy and study the relationship between them has gained a lot of attention in the study of dynamical systems.

Topological pressure is a generalization of topological entropy for a dynamical system. The notion was first introduced by Ruelle [26] in 1973 for an expansive dynamical system and later by Walters [29] for the general case. The variational principle formulated by Walters can be stated precisely as follows: Let (X,T) be a topological dynamical system, where X is a compact metric space and $T : X \to X$ is a continuous map, and $f : X \to \mathbb{R}$ is a continuous function. Let $P(T, f)$ denote the topological pressure of f (see [30]). Then

$$P(T,f) = \sup \left\{ h_\mu(T) + \int f \, \mathrm{d}\mu : \mu \in \mathcal{M}(X,T) \right\}, \tag{1.1}$$

where $\mathcal{M}(X,T)$ denotes the spaces of all T-invariant Borel probability measures on X and $h_\mu(T)$ denotes the measure-theoretic entropy of μ.

The theory related to the topological pressure, variational principle and equilibrium states plays a fundamental role in statistical mechanics, ergodic theory and dynamical systems (see, e.g., the books [5, 18, 27, 30]). Since the works of Bowen [6] and Ruelle [28], the topological pressure has become a basic tool in the dimension theory related to dynamical systems. In 1984, Pesin and Pitskel [25] defined the topological pressure of additive potentials for non-compact subsets of compact metric spaces and proved the variational principle under some supplementary conditions. In 1988, Falconer [8] considered the thermodynamic formalism for sub-additive potentials for mixing repellers. He proved the variational principle for the topological pressure under some Lipschitz conditions and bounded distortion assumptions on the sub-additive potentials. In 1996, Barreira [2] extended the work of Pesin and

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Pitskel. He defined the topological pressure for an arbitrary sequence of continuous functions on an arbitrary subset of compact metric spaces, and proved the variational principle under a strong convergence assumption on the potentials. In 2008, Y. Cao, D. Feng and W. Huang [7] generalized Ruelle and Walters's results to sub-additive potentials in general compact dynamical systems.

Since notions of entropy pairs were introduced in both topological [3] and measure-theoretic system [4], much attention has been paid to the study the local variational principle of entropy. Recently, Kerr and Li introduced various notions of independence and give a uniform treatment of entropy pairs and sequence entropy pairs [19, 20]. An overview of local entropy theory can see the survey paper [10]. In 2007, to study the local variational principle of topological pressure, W. Huang and Y. Yi [16] introduced a new definition of topological pressure for open covers. They proved a local variational principle for topological pressure for any given open cover.

In this paper, we generalize Huang-Yi's results to dynamical systems acting by a countable discrete amenable group. Let (X, G) be an amenable group action dynamical system. We define the local topological pressure for sub-additive potentials $\mathcal{F} = \{f_E\}_{E\in\mathcal{F}(G)}$ and set up a local variational principle between the topological pressure and measure-theoretical entropies.

Now we formulate our results. Throughout the paper, we let (X, G) be a G-system, where G is a countable discrete amenable group and X is a compact metric space. A **sub-additive potential on** (X, G) is a collection $\mathcal{F} = \{f_E\}_{E\in\mathcal{F}(G)}$ of continuous real valued function on X satisfies the following conditions:

(C1) $f_{E\cup F}(x) \leq f_E(x) + f_F(x)$ for all $x \in X$ and all $E \cap F = \emptyset, E, F \in \mathcal{F}(G)$;

(C2) $f_{Eg}(x) = f_E(gx)$ for all $x \in X$, $g \in G$ and $E \in \mathcal{F}(G)$;

(C3) $C = \sup\limits_{E\in F(X)} \sup\limits_{x\in X, g\in G} (f_E(x) - f_{E\cup\{g\}}(x)) < \infty$.

As a main result, we obtain the following local variational principle.

Theorem 1.1. (Local variational principle) *Let (X, G) be a G-system, $\mathcal{U} \in \mathcal{C}_X^o$ and $\mathcal{F} = \{f_E\}_{E\in\mathcal{F}(G)}$ a sub-additive potential on (X, G). Then*

$$P(G, \mathcal{F}; \mathcal{U}) = \sup_{\mu\in\mathcal{M}(X,G)} \{h_\mu(G, \mathcal{U}) + \mathcal{F}_*(\mu)\} \tag{1.2}$$

and the supremum can be attained in $\mathcal{M}^e(X, G)$, if one of the following conditions holds:

(1) *G is an Abelian group;*

(2) *$\mathcal{F}$ is strongly sub-additive, i.e. $f_{E\cup F} + f_{E\cap F} \leq f_E + f_F$ for all $E, F \in F(G)$.*

In particular, if f is a continuous function on X, then

$$\mathcal{F} = \left\{ f_E = \sum_{g\in E} f \circ g : E \in \mathcal{F}(G) \right\}$$

satisfies the condition (2) in Theorem 1.1. In this case, write $P(G, f; \mathcal{U}) = P(G, \mathcal{F}; \mathcal{U})$, we can get

Corollary 1.2. *Let (X,G) be a G-system, $\mathcal{U}\in\mathcal{C}_X^o$ and $f\in C(X)$, then*
$$P(G,f;\mathcal{U})=\sup_{\mu\in\mathcal{M}(X,G)}\left\{h_\mu(G,\mathcal{U})+\int_X f\,\mathrm{d}\mu\right\}$$
and the supremum can be attained in $\mathcal{M}^e(X,G)$.

The paper is organized as follows: in section 2, we recall some knowledge about amenable group, and the definition and basic properties of local measure-theoretic entropy for amenable group action. Moreover, we introduce the local pressure for a sub-additive potential. In section 3, we provide some useful lemmas and prove Theorem 1.1. In section 4, we give a nontrivial example of sub-additive potential.

2. Pressure of an Amenable Group Action

2.1. **Backgrounds of a countable discrete amenable group.** Let G be a countable discrete infinite group and $\mathcal{F}(G)$ the set of all finite non-empty subsets of G. A **tile** $T\subseteq G$ is a finite subset that has a collection of right translates that partitions G, i.e., there is a set $C\subseteq G$ of **tiling centers** such that $\{Tc:c\in C\}$ form a disjoint family whose union TC is all of G. Note that $T\in\mathcal{F}(G)$ is a tile of G if and only if any $A\in\mathcal{F}(G)$ can be covered by disjoint right translates of T.

A group G is said to be **amenable** if for all $\epsilon>0$ and all $K\in\mathcal{F}(G)$, there exists $F\in\mathcal{F}(G)$ such that
$$\frac{|F\triangle KF|}{|F|}<\epsilon.$$
Observe that a countable group is amenable if and only if there is a sequence $\{F_n\}_{n\in\mathbb{N}}\subseteq\mathcal{F}(G)$ such that
$$\lim_{n\to\infty}\frac{|KF_n\triangle F_n|}{|F_n|}=0$$
for all $K\in\mathcal{F}(G)$. Such a sequence is called a **Fϕlner sequence** of G (see [9]). For a more complete description of this class of groups see, for example, [13] or [24].

It is well known that the class of amenable groups contains all finite groups, Abelian groups, it is closed by taking subgroups, quotients, extensions and inductive limits. All finitely generated groups of subexponential growth are amenable. A basic example of a nonamenable group is the free group of rank 2.

Cyclic groups have Fϕlner sequences of tiling sets, and it can build up form them to show that all solvable groups, finite extensions thereof, increasing unions, etc., in brief the so-called class of **elementary amenable groups**, all have Fϕlner sequences of tiling sets. In particular, all Abelian group have tiling Fϕlner sequences. Unfortunately it is an open problem that whether all countable discrete amenable groups have Fϕlner sequences of tiling sets [23].

Let $f:\mathcal{F}(G)\to\mathbb{R}$ be a function. We say that f is
(i) **monotone**, if $f(E)\le f(F)$ for any $E,F\in\mathcal{F}(G)$ with $E\subseteq F$;
(ii) **non-negative**, if $f(F)\ge 0$ for any $F\in\mathcal{F}(G)$;
(iii) **G-invariant**, if $f(Fg)=f(F)$ for any $F\in\mathcal{F}(G)$ and $g\in G$;
(iv) **sub-additive**, if $f(E\cup F)\le f(E)+f(F)$ for any $E,F\in\mathcal{F}(G)$;

(v) **strongly sub-additive**, if $f(E \cup F) + f(E \cap F) \leq f(E) + f(F)$ for any $E, F \in \mathcal{F}(G)$.

The following limit theorem for invariant sub-additive functions on finite subsets of amenable groups is due to Ornstein and Weiss (see [14, 22, 23]). It plays a central role in the definition of some dynamical invariants such as topological entropy and measure-theoretic entropy.

Lemma 2.1. (Ornstein-Weiss) *Let G be a countable amenable group. Let $f : \mathcal{F}(G) \to \mathbb{R}$ be a monotone non-negative G-invariant sub-additive function. Then there is a real number $\lambda = \lambda(G, f) \geqslant 0$ dependent only on G and f such that*

$$\lim_{n\to\infty} \frac{f(F_n)}{|F_n|} = \lambda$$

for all Fϕlner sequence $\{F_n\}_{n\in\mathbb{N}}$ of G.

Remark 2.2. (1) If f is also strongly sub-additive, then

$$\lim_{n\to\infty} \frac{f(F_n)}{|F_n|} = \inf_{F\in F(G)} \frac{f(F)}{|F|}.$$

(2) If G admits a Fϕlner sequence $\{F_n\}_{n\in\mathbb{N}}$ of tiling sets, then

$$\lim_{n\to\infty} \frac{f(F_n)}{|F_n|} = \inf_{n\in\mathbb{N}} \frac{f(F_n)}{|F_n|},$$

and the value of the limits is independent of the choice of such a Fϕlner sequence. For details can see [31].

2.2. **Topological pressure for sub-additive potentials.** Let (X, G) be a G-system. Denote by $\mathcal{B}_X$ the collection of all Borel subsets of X. Recall that a **cover** of X is a family of Borel subsets of X whose union is X. An **open cover** is one that consists of open sets. A **partition** of X is a cover of X consisting of pairwise disjoint sets. We denote the set of finite covers, finite open covers and finite partition of X by $\mathcal{C}_X$, $\mathcal{C}_X^o$ and $\mathcal{P}_X$, respectively. Given two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, $\mathcal{U}$ is said to be **finer** than $\mathcal{V}$ (denoted by $\mathcal{U} \succeq \mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Given $F \in \mathcal{F}(G)$ and $\mathcal{U} \in \mathcal{C}_X$, set $\mathcal{U}_F = \bigvee_{g\in F} g^{-1}\mathcal{U}$ (letting $\mathcal{U}_\emptyset = \{X\}$).

We now define the topological pressure of sub-additive potential $\mathcal{F}$ relative an open cover. For $E \in \mathcal{F}(G)$ and $\mathcal{U} \in \mathcal{C}_X^o$, we define

$$P_E(G, \mathcal{F}; \mathcal{U}) := \inf \left\{ \sum_{V\in\mathcal{V}} \sup_{x\in V} \mathrm{e}^{f_E(x)} : \mathcal{V} \in \mathcal{C}_X \text{ and } \mathcal{V} \succeq \mathcal{U}_E \right\}.$$

For $\mathcal{V} \in \mathcal{C}_X$, we let α be the Borel partition generated by $\mathcal{V}$ and define

$$\mathcal{P}^*(\mathcal{V}) = \left\{ \beta \in \mathcal{P}_X : \begin{array}{l} \beta \succeq \mathcal{V} \text{ and each atom of } \beta \text{ is the union of} \\ \text{some atoms of } \alpha. \end{array} \right\}. \tag{2.1}$$

Note that $\mathcal{P}^*(\mathcal{V})$ is a finite set. Following the idea of Huang-Yi (see [16, Lemma 2.1]), we have the following result.

Lemma 2.3. *For $E \in \mathcal{F}(G)$ and $\mathcal{U} \in \mathcal{C}_X$, we have*

$$P_E(T,\mathcal{F};\mathcal{U}) = \min\left\{\sum_{B\in\beta}\sup_{x\in B} \mathrm{e}^{f_E(x)} : \beta \in \mathcal{P}^*(\mathcal{U}_E)\right\}. \tag{2.2}$$

Lemma 2.4. *The following hold:*

(1) $K = \sup\left\{\frac{|f_E(x)|}{|E|} : x \in X, E \in F(G)\right\} < \infty$*;*

(2) *Set $\mathcal{G} = \{f_E(x) + C|E| : E \in F(G), f_E \in \mathcal{F}\}$, then $\mathcal{G}$ is a monotone non-negative sub-additive function. If $\mathcal{F}$ is strongly sub-additive, then $\mathcal{G}$ is also strongly sub-additive.*

Proof. It easily follows from conditions (C1), (C2) and (C3). □

It is not hard to see that $E \in \mathcal{F}(G) \mapsto \log P_E(G,\mathcal{G},\mathcal{U})$ is a monotone non-negative G-invariant sub-additive function. By Lemma 2.1,

$$\lim_{n\to\infty}\frac{1}{|F_n|}\log P_{F_n}(G,\mathcal{F};\mathcal{U}) = \lim_{n\to\infty}\frac{1}{|F_n|}\log P_{F_n}(G,\mathcal{G};\mathcal{U}) - C$$

is independence of the choice of the Fϕlner sequence $\{F_n\}_{n\in\mathbb{N}}$. Define the **topological pressure of $\mathcal{F}$ relative to $\mathcal{U}$** as

$$P(G,\mathcal{F};\mathcal{U}) := \lim_{n\to\infty}\frac{1}{|F_n|}\log P_{F_n}(G,\mathcal{F};\mathcal{U}), \tag{2.3}$$

where $\{F_n\}_{n\in\mathbb{N}}$ is a Fϕlner sequence of G. The **topological pressure of $\mathcal{F}$** is defined by

$$P(G,\mathcal{F}) := \sup_{\mathcal{U}\in\mathcal{C}_X^o} P(G,\mathcal{F};\mathcal{U}) \tag{2.4}$$

For a G-invariant Borel probability measure μ, denote

$$\mathcal{F}_*(\mu) := \lim_{n\to\infty}\frac{1}{|F_n|}\int f_{F_n}\,\mathrm{d}\mu,$$

where $\{F_n\}_{n\in\mathbb{N}}$ is a Fϕlner sequence. The existence of the above limit follows from conditions (C1) and (C2). We call $\mathcal{F}_*(\mu)$ the **Lyapunov exponent of $\mathcal{F}$ with respect to μ.**

2.3. **Measure-theoretic entropy.** Recall the basic definitions (see [17] for details). Let $\mathcal{M}(X)$, $\mathcal{M}(X,G)$ and $\mathcal{M}^e(X,G)$ be the sets of all Borel probability measures, G-invariant Borel probability measures on X and G-invariant ergodic measures, on X, respectively. Note that amenability of G ensures that $\mathcal{M}(X,G) \neq \emptyset$ and both $\mathcal{M}(X)$ and $\mathcal{M}(X,G)$ are convex compact metric spaces when endowed with the weak*-topology; $\mathcal{M}^e(X,G)$ is a G_δ subset of $\mathcal{M}(X,G)$.

Given $\alpha,\beta \in \mathcal{P}_X$ and $\mu \in \mathcal{M}(X)$, define

$$H_\mu(\alpha) = \sum_{A\in\alpha} -\mu(A)\log\mu(A) \quad \text{and} \quad H_\mu(\alpha|\beta) = H_\mu(\alpha\vee\beta) - H_\mu(\beta).$$

One standard fact is that $H_\mu(\alpha|\beta)$ increases with respect to α and decreases with respect to β. When $\mu \in \mathcal{M}(X,G)$, it is not hard to see that $F \in \mathcal{F}(G) \mapsto H_\mu(\alpha_F)$

is a monotone non-negative G-invariant sub-additive function for a given $\alpha \in \mathcal{P}_X$. The **measure-theoretic entropy of μ relative to α** is defined by

$$h_\mu(G,\alpha) = \lim_{n\to\infty} \frac{1}{|F_n|} H_\mu(\alpha_{F_n}) = \inf_{F\in F(G)} \frac{1}{|F|} H_\mu(\alpha_F), \tag{2.5}$$

where $\{F_n\}_{n\in\mathbb{N}}$ is a Fϕlner sequence of G. The last identity follows from the fact that $H_\mu(\alpha_F)$ is strongly sub-additive (see [17, Lemma 3.1]). The **measure-theoretic entropy of μ** is defined by

$$h_\mu(G,X) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(G,\alpha). \tag{2.6}$$

For a given $\mathcal{U} \in \mathcal{C}_X$, W. Huang, X. Ye and G. Zhang (see [17]) introduced the following two types of **measure-theoretic entropies relative to $\mathcal{U}$** as

$$h_\mu^-(G,\mathcal{U}) := \lim_{n\to\infty} \frac{1}{|F_n|} H_\mu(\mathcal{U}_{F_n}) \quad \text{and} \quad h_\mu^+(G,\mathcal{U}) := \inf_{\alpha\succeq\mathcal{U},\alpha\in\mathcal{P}_X} h_\mu(G,\alpha),$$

where

$$H_\mu(\mathcal{U}) := \inf_{\alpha\succeq\mathcal{U},\alpha\in\mathcal{P}_X} H_\mu(\alpha).$$

Remark 2.5. (1) It is not hard to see that $h_\mu^-(G,\mathcal{U}) \le h_\mu^+(G,\mathcal{U})$. Moreover, Huang-Ye-Zhang (see [17, Theorem 4.14]) proved those two kinds of measure-theoretic entropy are equivalent, thus, we denote by

$$h_\mu(G,\mathcal{U}) = h_\mu^\pm(G,\mathcal{U}).$$

(2) For $\mu \in \mathcal{M}(X,G)$, the following holds (see [17]):

$$h_\mu(G,X) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h_\mu(G,\mathcal{U}).$$

Lemma 2.6. (Ergodic decomposition of local entropy, [17]**)** *Let $\mathcal{U} \in \mathcal{C}_X^o$ and $\mu \in \mathcal{M}(X,G)$. The local entropy function $h_\cdot(G,\mathcal{U})$ is upper semi-continuous and affine on $\mathcal{M}(X,G)$, and*

$$h_\mu(G,\mathcal{U}) = \int_{\mathcal{M}^e(X,G)} h_\theta(G,\mathcal{U})\,\mathrm{d}m(\theta),$$

where $\mu = \int_{\mathcal{M}^e(X,G)} \theta\,\mathrm{d}m(\theta)$ is the ergodic decomposition of μ.

3. A local variational principle of topological pressure

In this section, we mainly prove a local variational principle of topological pressure for sub-addition potentials.

3.1. **Some Lemmas.** Now we give some lemmas which are needed in our proof of Theorem 1.1. The first lemma is an obvious fact and we omit the detailed proof.

Lemma 3.1. *Let $T \in \mathcal{F}(G)$ be a tile of G and $\{F_n\}_{n\in\mathbb{N}}$ a Følner sequence. For each $n \in \mathbb{N}$, let C_n be the tiling center of F_n relative to T, i.e., $F_n \subseteq \bigsqcup_{c\in C_n} Tc$ and $Tc \cap F_n \neq \emptyset$ for all $c \in C_n$, then*

$$\lim_{n\to\infty} \frac{|TC_n|}{|F_n|} = 1.$$

Lemma 3.2. *Let $f:\mathcal{F}(G)\to\mathbb{R}$ be a non-negative monotone strongly sub-additive function, $m,k\in\mathbb{N}$, $E,F,B,E_1,\cdots,E_k\in\mathcal{F}(G)$. Then*

(1) *If $1_E(g)=\frac{1}{m}\sum_{i=1}^k 1_{E_i}(g)$ holds for each $g\in G$, then $f(E)\le\frac{1}{m}\sum_{i=1}^k f(E_i)$;*

(2) *If $K=\sup\{\frac{f(E)}{|E|}:E\in\mathcal{F}(G)\}<\infty$, then*

$$f(F)\le\sum_{g\in F}\frac{1}{|B|}f(Bg)+K\cdot|F\setminus A_{F,B}|,$$

where, $A_{F,B}=\{g\in G:B^{-1}g\subseteq F\}$.

Proof. (1) Clearly, $\bigcup_{i=1}^k E_i=E$. Set $\{A_1,\cdots,A_n\}=\bigvee_{i=1}^k\{E_i,E\setminus E_i\}$ (neglecting all empty elements). Set $K_0=\emptyset$, $K_i=\bigcup_{j=1}^i A_j$, $i=1,\cdots,n$. Then $\emptyset=K_0\subsetneq K_1\cdots\subsetneq K_n=E$. Note that if for some $i=1,\cdots,n$ and $j=1,\cdots,k$ with $E_j\cap(K_i\setminus K_{i-1})\neq\emptyset$, then $K_i\setminus K_{i-1}\subseteq E_j$, and so $K_i=K_{i-1}\cup(K_i\cap E_j)$. By strongly sub-additive of f, we have $f(K_i)+f(K_{i-1}\cap E_j)\le f(K_{i-1})+f(K_i\cap E_j)$, i.e.,

$$f(K_i)-f(K_{i-1})\le f(K_i\cap E_j)-f(K_{i-1}\cap E_j).$$

Now for each $i=1,\cdots,n$, we pick $k_i\in K_i\setminus K_{i-1}$, one has

$$\begin{aligned}
f(E) &= \sum_{i=1}^n\left(\frac{1}{m}\sum_{i=1}^k 1_{E_i}(k_i)\right)(f(K_i)-f(K_{i-1}))\\
&= \frac{1}{m}\sum_{j=1}^k\sum_{\substack{1\le i\le n\\ k_i\in E_j}}(f(K_i)-f(K_{i-1}))\\
&\le \frac{1}{m}\sum_{j=1}^k\sum_{\substack{1\le i\le n\\ k_i\in E_j}}(f(K_i\cap E_j)-f(K_{i-1}\cap E_j))\\
&\le \frac{1}{m}\sum_{j=1}^k\sum_{i=1}^n(f(K_i\cap E_j)-f(K_{i-1}\cap E_j))=\frac{1}{m}\sum_{j=1}^k f(E_j),
\end{aligned}$$

(2) Note that for each $l\in G$, we have $1_{\{h\in BF:B^{-1}h\subseteq F\}}(l)=\frac{1}{|B|}\sum_{g\in F}1_{\{h\in Bg:B^{-1}h\subseteq F\}}(l)$. Using (1), we can get

$$f\left(\{h\in BF:B^{-1}h\subseteq F\}\right)\le\frac{1}{|B|}\sum_{g\in F}f\left(\{h\in Bg:B^{-1}h\subseteq F\}\right)\le\frac{1}{|B|}\sum_{g\in F}f(Bg),$$

which implies

$$\begin{aligned}
f(F) &\le f\left(\{h\in BF:B^{-1}h\subseteq F\}\right)+f(F\setminus\{h\in BF:B^{-1}h\subseteq F\})\\
&\le \frac{1}{|B|}\sum_{g\in F}f(Bg)+|F\setminus\{h\in BF:B^{-1}h\subseteq F\}|\cdot K\\
&= \frac{1}{|B|}\sum_{g\in F}f(Bg)+K\cdot|F\setminus A_{F,B}|.
\end{aligned}$$

The lemma is proved. □

Lemma 3.3. *Let (X,G) be a zero-dimensional G-system, $\mu \in \mathcal{M}(X,G)$, $\mathcal{F}$ is a sub-additive potential and $\mathcal{U} \in \mathcal{C}_X^o$. Assume that for some $K \in \mathbb{N}$, $\{\alpha_l\}_{l=1}^K$ is a sequence of finite clopen (close and open) partitions of X which are finer that $\mathcal{U}$. Then for each $E \in \mathcal{F}(G)$, there is a finite subset B_E of X such that each atom of $(\alpha_l)_E$, $l = 1, \cdots, K$, contains at most one point of B_E, and $\sum_{x\in B_E} \mathrm{e}^{f_E(x)} \geq \frac{P_E(G,\mathcal{F},\mathcal{U})}{K}$.*

Proof. The proof follows completely from that of [16, Lemma 4.4] and is omitted. □

Let (X,G) and (Y,G) be two G-systems. A continuous map $\pi : X \to Y$ is called a **homomorphism** or a **factor map** from (X,G) to (Y,G) if it is onto and $\pi\circ g = g\circ\pi$ for all $g \in G$. (X,G) is called an **extension** of (Y,G) and (Y,G) is called a **factor** of (X,G). If π is also injective then it is called an **isomorphism**.

Lemma 3.4. *Let $\pi : (X,G) \to (Y,G)$ be a factor map, $\mathcal{F} = \{f_E\}_{E\in\mathcal{F}(G)} \subseteq C(Y)$ satisfies conditions (C1), (C2) and (C3). Let $\mu \in \mathcal{M}(X,G)$, $\nu = \pi\mu$, $\alpha \in \mathcal{P}_Y$ and $\mathcal{U} \in \mathcal{C}_Y^o$. Then*
(1) $h_\mu(G,\pi^{-1}(\alpha)) = h_\nu(G,\alpha)$;
(2) $P(G,\mathcal{F}\circ\pi;\pi^{-1}\mathcal{U}) = P(G,\mathcal{F};\mathcal{U})$;

Proof. (1) It is an obvious fact.

(2) Let $\{F_n\}_{n\in\mathbb{N}}$ be a Fϕlner sequence of G. Fix an $n \in \mathbb{N}$. If $\mathcal{V} \in \mathcal{C}_Y$ with $\mathcal{V} \succeq \mathcal{U}_{F_n}$, then $\pi^{-1}\mathcal{V} \in \mathcal{C}_X$ and $\pi^{-1}\mathcal{V} \succeq (\pi^{-1}\mathcal{U})_{F_n}$. Hence

$$\sum_{V\in\mathcal{V}} \sup_{y\in V} \mathrm{e}^{f_{F_n}(y)} = \sum_{V\in\mathcal{V}} \sup_{z\in\pi^{-1}V} \mathrm{e}^{f_{F_n}\circ\pi(z)} \geq P_{F_n}(G,\mathcal{F}\circ\pi;\pi^{-1}\mathcal{U}).$$

Since $\mathcal{V}$ is arbitrary, we have that $P_{F_n}(G,\mathcal{F};\mathcal{U}) \geq P_{F_n}(G,\mathcal{F}\circ\pi;\pi^{-1}\mathcal{U})$.

Conversely, we note that

$$P_{F_n}(G,\mathcal{F}\circ\pi;\pi^{-1}\mathcal{U}) = \min_{\beta\in\mathcal{P}^*((\pi^{-1}\mathcal{U})_{F_n})} \left\{ \sum_{B\in\beta} \sup_{x\in B} \mathrm{e}^{f_{F_n}(x)} \right\}.$$

Let $\beta \in \mathcal{P}^*((\pi^{-1}\mathcal{U})_{F_n})$, then $\pi(\beta) \in \mathcal{C}_Y$, $\pi(\beta) \succeq \mathcal{U}_{F_n}$, and thus

$$\sum_{B\in\beta} \sup_{x\in B} \mathrm{e}^{f_{F_n}\circ\pi(x)} = \sum_{B\in\beta} \sup_{y\in\pi(B)} \mathrm{e}^{f_{F_n}(y)} \geq P_{F_n}(G,\mathcal{F};\mathcal{U}).$$

Since β is arbitrary, $P_{F_n}(G,\mathcal{F}\circ\pi;\pi^{-1}\mathcal{U}) \geq P_{F_n}(G,\mathcal{F};\mathcal{U})$.

Above all, $P_{F_n}(G,\mathcal{F}\circ\pi;\pi^{-1}\circ\mathcal{U}) = P_{F_n}(G,\mathcal{F};\mathcal{U})$ for each $n \in \mathbb{N}$, from which the lemma follows. □

For a fixed $\mathcal{U} = \{U_1, \cdots, U_M\} \in \mathcal{C}_X^o$, we let

$$\mathcal{U}^* = \{\{A_1, \cdots, A_M\} \in \mathcal{P}_X : A_m \subseteq U_m : 1 \leq m \leq M\}.$$

The following lemma will be used in the computation of $H_\mu(\mathcal{U})$ and $h_\mu(T,\mathcal{U})$ (see [15, Lemma 2] for detail).

Lemma 3.5. *Let $H : \mathcal{P}_X \to \mathbb{R}$ be monotone in the sense that $H(\alpha) \geq H(\beta)$ whenever $\alpha \succeq \beta$. Then*

$$\inf_{\alpha\in\mathcal{P}_X,\alpha\succeq\mathcal{U}} H(\alpha) = \inf_{\alpha\in\mathcal{U}^*} H(\alpha).$$

Lemma 3.6. *Let (X,G) be G-system, where G is a Abelian group. Suppose $\{\nu_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{M}(X)$ and $\{F_n\}_{n=1}^{\infty}$ is a tiling Fϕlner sequence of G. We form the new sequence $\{\mu_n\}_{n=1}^{\infty}$ by $\mu_n = \frac{1}{|F_n|}\sum_{g\in F_n} g\nu_n$. Assume that μ_{n_i} converges to μ in $\mathcal{M}(X)$ for some subsequence $\{n_i\}$ of natural numbers. Then $\mu \in \mathcal{M}(X,G)$, and moreover*

$$\limsup_{i\to\infty} \frac{1}{|F_{n_i}|}\int f_{F_{n_i}}\,\mathrm{d}\nu_{n_i} \le \mathcal{F}_*(\mu). \tag{3.1}$$

Proof. The statement $\mu \in \mathcal{M}(X,G)$ is well-known. Now we show the desired inequality. Fix $k\in\mathbb{N}$. Since F_k is a tile of G, let C_n is a tiling center of F_n relative to F_k, i.e.,

$$\bigsqcup_{c\in C_n} F_k c \supseteq F_n \quad\text{and}\quad F_k c\cap F_n \neq \emptyset,\ \forall\ c\in C_n, n\in\mathbb{N}. \tag{3.2}$$

By Lemma 3.1, for each $\epsilon>0$, when n large enough, we have

$$|F_kC_n| \le |(1+\epsilon)|F_n| \quad\text{and}\quad |F_kC_n\setminus F_n| \le \epsilon|F_n|. \tag{3.3}$$

Without loss of generality, we can assume $\mathcal{F}$ is nonnegative monotone, then

$$f_{F_na}(x) \le f_{\bigsqcup_{c\in C_n}F_kca}(x) \le \sum_{c\in C_n} f_{F_kca}(x), \quad \forall a\in F_k.$$

Set $g_{F_n}(x) = \frac{1}{|F_k|}\sum_{a\in F_k} f_{F_na}(x)$. Since G is a Abelian group, we have

$$g_{F_n}(x) \le \frac{1}{|F_k|}\sum_{a\in F_k, c\in C_n} f_{F_kca}(x) = \frac{1}{|F_k|}\sum_{g\in F_kC_n} f_{F_kg}(x).$$

Moreover, by (3.3), we can get

$$\begin{aligned}
\frac{1}{|F_n|}\int_X g_{F_n}(x)d\nu_n(x) &\le \frac{1}{|F_n||F_k|}\int_X \sum_{g\in F_kC_n} f_{F_kg}(x)d\nu_n(x)\\
&= \frac{|F_kC_n|}{|F_n||F_k|}\int_X f_{F_k}(x)d\widetilde{\mu}_n(x)\\
&\le \frac{1+\epsilon}{|F_k|}\int_X f_{F_k}(x)d\widetilde{\mu}_n(x),
\end{aligned}$$

where $\widetilde{\mu}_n = \frac{1}{|F_kC_n|}\sum_{g\in F_kC_n} g\nu_n$. To complete the lemma, it suffices to show the following two claims hold.

Claim 1. $\displaystyle\lim_{n\to\infty}\frac{1}{|F_n|}\int_X |f_{F_n}(x) - g_{F_n}(x)|\,\mathrm{d}\nu_n(x) = 0;$

Proof of Claim 1. Since $\mathcal{F}$ is nonnegative, monotone and sub-additive, for each $a\in F_k$,

$$f_{F_n}(x) \le f_{F_na}(x) + f_{F_n\setminus F_na}(x) \le f_{F_na}(x) + K\cdot|F_n\setminus F_na|.$$

By symmetry, $|f_{F_n}(x) - f_{F_na}(x)| \le K\cdot|F_n \triangle F_na|$. Thus,

$$|f_{F_n}(x) - g_{F_n}(x)| = \left|\frac{1}{|F_k|}\sum_{a\in F_k} f_{F_n}(x) - f_{F_na}(x)\right| \le \frac{K}{|F_k|}\sum_{a\in F_k}|F_n\triangle F_na|.$$

Therefore,

$$\frac{1}{|F_n|}\int_X |f_{F_n}(x)-g_{F_n}(x)|\,\mathrm{d}\nu_n \leq \frac{K}{|F_k|}\sum_{a\in F_k}\frac{|F_n \triangle F_n a|}{|F_n|} \to 0 \quad (n\to\infty)$$

This complete the proof of Claim 1. □

Claim 2. with the weak*-topology, $\widetilde{\mu}_{n_i} \to \mu$.

Proof of Claim 2. It suffices to show that for each $f \in C(X)$,

$$\left|\int_X f\,\mathrm{d}\mu_n - \int_X f\,\mathrm{d}\widetilde{\mu}_n\right| \to 0 \quad (n\to\infty) \tag{3.4}$$

By Lemma 3.1, $\lim_{n\to\infty}\frac{|F_kC_n|}{|F_n|}=1$. So

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_X f\,\mathrm{d}\mu_n - \int_X f\,\mathrm{d}\widetilde{\mu}_n\right| \\
=\ &\lim_{n\to\infty}\left|\frac{1}{|F_n|}\sum_{g\in F_n}\int_X f(gx)\,\mathrm{d}\nu_n(x) - \frac{1}{|F_kC_n|}\sum_{g\in F_kC_n}\int_X f(gx)\,\mathrm{d}\nu_n(x)\right| \\
=\ &\lim_{n\to\infty}\left|\frac{1}{|F_n|}\sum_{g\in F_n}\int_X f(gx)\,\mathrm{d}\nu_n(x) - \frac{1}{|F_n|}\sum_{g\in F_kC_n}\int_X f(gx)\,\mathrm{d}\nu_n(x)\right| \\
\leqslant\ &\lim_{n\to\infty}\frac{|F_kC_n\setminus F_n|}{|F_n|}\cdot\|f\| = 0
\end{aligned}$$

This complete the proof of Claim 2. □

The following lemma is well known (see [30, §9] for a proof).

Lemma 3.7. *Let $a_1, a_2, \cdots, a_k$ be given real numbers. If $p_i \geq 0, i=1,2,\cdots,k$ and $\sum_{i=1}^k p_i = 1$, then*

$$\sum_{i=1}^k p_i\left(a_i - \log p_i\right) \leq \log\left(\sum_{i=1}^k \mathrm{e}^{a_i}\right), \tag{3.5}$$

and equality holds if and only if $p_i = \frac{\mathrm{e}^{a_i}}{\sum_{i=j}^k \mathrm{e}^{a_j}}$ for all $i=1,2,\cdots,k$.

3.2. **Proof of Theorem 1.1.** In this section we give the proof of Theorem 1.1.

Proof of Theorem 1.1 We divide the proof into three small steps:

Step 1. $P(G,\mathcal{F};\mathcal{U}) \geq h_\mu(T,\mathcal{U}) + \mathcal{F}_*(\mu)$ for all $\mu \in \mathcal{M}(X,G)$.

Let $\mu \in \mathcal{M}(X,G)$ and $\{F_n\}_{n\in\mathbb{N}}$ a Fϕlner sequence of G. By (2.2), there exists a finite partition $\beta \in \mathcal{P}^*(\mathcal{U}_{F_n})$ such that

$$P_{F_n}(G,\mathcal{F},\mathcal{U}) = \sum_{B\in\beta}\sup_{x\in B}\mathrm{e}^{f_{F_n}(x)}.$$

It follows from Lemma 3.7 that

$$\begin{aligned}\log P_{F_n}(G,\mathcal{F},\mathcal{U}) &= \log\left(\sum_{B\in\beta}\sup_{x\in B} \mathrm{e}^{f_{F_n}(x)}\right)\\ &\geq \sum_{B\in\beta}\mu(B)\left(\sup_{x\in B} f_{F_n}(x)-\log\mu(B)\right)\ \text{(by (3.5))}\\ &= H_\mu(\beta)+\sum_{B\in\beta}\sup_{x\in B} f_{F_n}(x)\cdot\mu(B)\\ &\geqslant H_\mu(\mathcal{U}_{F_n})+\int_X f_{F_n}\,\mathrm{d}\mu\end{aligned}$$

The proof of step 1 is complete by dividing the above by $|F_n|$ then passing the limit $n\to\infty$.

Step 2. If (X,G) is a zero-dimensional G-system, then there exists a $\mu\in\mathcal{M}(X,G)$ such that

$$P(G,\mathcal{F};\mathcal{U})\leq h_\mu(G,\mathcal{U})+\mathcal{F}_*(\mu). \tag{3.6}$$

Let $\mathcal{U}=\{U_1,U_2,\cdots,U_d\}$ and define

$$\mathcal{U}^*=\{\alpha\in\mathcal{P}_X:\alpha=\{A_1,A_2,\cdots,A_d\},A_m\subseteq U_m,m=1,2,\cdots,d\}.$$

Since X is zero-dimensional, the family of partitions in $\mathcal{U}^*$, which are finer than $\mathcal{U}$ and consist of clopen (close and open) sets, is countable. We let $\{\alpha_l\}_{l\in\mathbb{N}}$ denote an enumeration of this family.

Let $\{F_n\}_{n\in\mathbb{N}}$ be a Fϕlner sequence of G with $|F_n|\geq n$ for each $n\in\mathbb{N}$. By Lemma 3.3, for each $n\in\mathbb{N}$, there exists a finite subset B_n of X such that

$$\sum_{x\in B_{F_n}}\mathrm{e}^{f_{F_n}(x)}\geq\frac{P_{F_n}(G,\mathcal{F};\mathcal{U})}{n}, \tag{3.7}$$

and each atom of $(\alpha_l)_{F_n}$ contains at most one point of B_n, for each $l=1,\cdots,n$. Let

$$\nu_n=\sum_{x\in B_n}\lambda_n(x)\delta_x\quad\text{and}\quad\mu_n=\frac{1}{|F_n|}\sum_{g\in F_n}g\nu_n,$$

where $\lambda_n(x)=\frac{\mathrm{e}^{f_{F_n}(x)}}{\sum_{y\in B_n}\mathrm{e}^{f_{F_n}(x)}}$ for $x\in B_n$. Since $\mathcal{M}(X,G)$ is compact we can choose a subsequence $\{n_i\}\subseteq\mathbb{N}$ such that $\mu_{n_i}\to\mu$ in the weak*-topology of $\mathcal{M}(X)$. It is easy to check $\mu\in\mathcal{M}(X,G)$. We wish to show that μ satisfies (3.6). By Lemma 3.5 and the fact that

$$h^+_\mu(G,\mathcal{U})=\inf_{\beta\in\mathcal{U}^*}h_\mu(G,\beta)=\inf_{l\in\mathbb{N}}h_\mu(G,\alpha_l),$$

it is sufficient to show that for each $l\in\mathbb{N}$

$$P(G,\mathcal{F};\mathcal{U})\leq h_\mu(G,\alpha_l)+\mathcal{F}_*(\mu). \tag{3.8}$$

Fix $l \in \mathbb{N}$. For each $n > l$, we know from the construction of B_n that each atom of $(\alpha_l)_{F_n}$ contains at most one point of B_n, and

$$\sum_{x\in B_n} -\lambda_n(x)\log\lambda_n(x) = \sum_{x\in B_n} -\nu_n(\{x\})\log\nu_n(\{x\}) = H_{\nu_n}((\alpha_l)_{F_n}). \tag{3.9}$$

Moreover, it follows from (3.7), (3.9) that

$$\begin{aligned}\log P_{F_n}(G,\mathcal{F};\mathcal{U}) - \log n &\le \log\left(\sum_{x\in B_n} \mathrm{e}^{f_{F_n}(x)}\right) = \sum_{x\in B_n}\lambda_n(x)(f_{F_n}(x) - \log\lambda_n(x)) \\ &= H_{\nu_n}((\alpha_l)_{F_n}) + \sum_{x\in B_n}\lambda_n(x) f_{F_n}(x) \\ &= H_{\nu_n}((\alpha_l)_{F_n}) + \int_X f_{F_n}(x)\,\mathrm{d}\nu_n(x).\end{aligned}$$

Hence,

$$\log P_{F_n}(G,\mathcal{F};\mathcal{U}) - \log n \le H_{\nu_n}((\alpha_l)_{F_n}) + \int_X f_{F_n}(x)\,\mathrm{d}\nu_n(x) \tag{3.10}$$

Without loss of generality, we can assume $\mathcal{F}$ is nonnegative, monotone and sub-additive.

Case 1. G is Abelian group. We can assume $\{F_n\}_{n\in\mathbb{N}}$ is a tiling Fϕlner sequence. Since $E \in F(G) \mapsto H_{\nu_n}((\alpha_l)_E)$ is a nonnegative, monotone and strongly sub-additive function, it follows from Lemma 3.2 that for each $B \in F(G)$, one has

$$\begin{aligned}\frac{1}{|F_n|}H_{\nu_n}((\alpha_l)_{F_n}) &\le \frac{1}{|F_n|}\sum_{g\in F_n}\frac{1}{|B|}H_{\nu_n}((\alpha_l)_{Bg}) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|}\cdot\log|\alpha_l| \\ &= \frac{1}{|B|}\frac{1}{|F_n|}\sum_{g\in F_n} H_{g\nu_n}((\alpha_l)_B) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|}\cdot\log d \\ &\le \frac{1}{|B|}H_{\mu_n}((\alpha_l)_B) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|}\cdot\log d.\end{aligned} \tag{3.11}$$

Set $B_1 = B^{-1}\cup\{e_G\}$. Note that for each $\delta > 0$, F_n is (B_1,δ)-invariant if n is large enough and

$$F_n \setminus A_{F_n,B} = F_n \cap B(G\setminus F_n) \subseteq (B_1)^{-1}F_n \cap (B_1)^{-1}(G\setminus F_n) = B(F_n, B_1).$$

Letting $n \to \infty$, we can get

$$\lim_{n\to\infty}\frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \le \lim_{n\to\infty}\frac{|B(F_n,B_1)|}{|F_n|} = 0. \tag{3.12}$$

Hence, combining Lemma 3.6, (3.10), (3.11) and (3.12) we obtain

$$\begin{aligned} P(G,\mathcal{F};\mathcal{U}) &= \lim_{i\to\infty} \frac{\log P_{F_{n_i}}(G,\mathcal{F};\mathcal{U})}{|F_{n_i}|} \\ &\le \limsup_{i\to\infty} \left(\frac{1}{|F_{n_i}|} H_{\nu_{n_i}}((\alpha_l)_{F_{n_i}}) + \frac{\log n_i}{|F_{n_i}|} + \frac{1}{|F_{n_i}|} \int_X f_{F_{n_i}}(x)\,\mathrm{d}\nu_{n_i}(x) \right) \\ &\le \frac{1}{|B|} H_\mu((\alpha_l)_B) + \mathcal{F}_*(\mu). \end{aligned}$$

By arbitrary of $B \in \mathcal{F}(G)$, (3.8) holds.

Case 2. $\mathcal{F}$ is strongly sub-additive. Now $E \in F(G) \mapsto \int_X f_E(x)\,\mathrm{d}\nu_n(x)$ is a nonnegative monotone strongly sub-additive function. By Lemma 3.2, for each $B \in \mathcal{F}(G)$, one has

$$\begin{aligned} \frac{1}{|F_n|} \int_X f_{F_n}(x)\,\mathrm{d}\nu_n(x) &\le \frac{1}{|F_n|} \sum_{g\in F_n} \frac{1}{|B|} \int_X f_{Bg}(x)\,\mathrm{d}\nu_n(x) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \cdot K \\ &= \frac{1}{|B|} \int_X f_B(x)\,\mathrm{d}\mu_n(x) + \frac{|F_n \setminus A_{F_n,B}|}{|F_n|} \cdot K \end{aligned} \tag{3.13}$$

Combining (3.10), (3.11), (3.12) and (3.13), we have

$$\begin{aligned} P(G,\mathcal{F};\mathcal{U}) &= \lim_{i\to\infty} \frac{\log P_{F_{n_i}}(G,\mathcal{F};\mathcal{U})}{|F_{n_i}|} \\ &\le \limsup_{i\to\infty} \left(\frac{1}{|F_{n_i}|} H_{\nu_{n_i}}((\alpha_l)_{F_{n_i}}) + \frac{\log n_i}{|F_{n_i}|} + \frac{1}{|F_{n_i}|} \int_X f_{F_{n_i}}(x)\,\mathrm{d}\nu_{n_i}(x) \right) \\ &\le \frac{1}{|B|} H_\mu((\alpha_l)_B) + \frac{1}{|B|} \int_X f_B(x)\,\mathrm{d}\mu(x) \end{aligned}$$

By arbitrary of $B \in \mathcal{F}(G)$ and (2) in Remark 2.2, (3.8) holds.

Step 3. For G-system (X,G), there exists a $\mu \in \mathcal{M}(X,G)$ such that (3.6) holds. It is well known that there exists factor map $\pi : (Z,G) \to (X,G)$, where (Z,G) is a zero-dimensional G-systems (see [17], for example). Using Step 2, there is $\nu \in \mathcal{M}(Z,G)$ such that

$$P(G,\mathcal{F}\circ\pi;\pi^{-1}(\mathcal{U})) \leqslant h_\nu(G,\pi^{-1}\mathcal{U}) + (\mathcal{F}\circ\pi)_*(\nu).$$

Let $\mu = \pi_*(\nu)$. By Lemma 3.4, we can get

$$\begin{aligned} h_\mu(G,\mathcal{U}) + \mathcal{F}_*(\mu) &= \inf_{\alpha\in\mathcal{P}_X,\alpha\succeq\mathcal{U}} (h_\mu(G,\alpha) + \mathcal{F}*(\mu)) \\ &= \inf_{\alpha\in\mathcal{P}_X,\alpha\succeq\mathcal{U}} \big(h_\nu(G,\pi^{-1}(\alpha)) + (\mathcal{F}\circ\pi)*(\nu)\big) \\ &\ge h_\nu(G,\pi^{-1}\mathcal{U}) + (\mathcal{F}\circ\pi)*(\nu) \\ &\ge P(G,\mathcal{F}\circ\pi;\pi^{-1}(\mathcal{U})) = P(G,\mathcal{F};\mathcal{U}). \end{aligned}$$

We will show the supremum of (1.2) can be attained in $\mathcal{M}^e(X,G)$. Let $\mu = \int_{\mathcal{M}^e(X,G)} \theta \,\mathrm{d}m(\theta)$ be the ergodic decomposition of μ. Note that $\theta \mapsto \mathcal{F}_*(\theta)$ is m-measurable and $\sup\{\frac{|f_{F_n}(x)|}{|F_n|} : x \in X, F_n \in \mathcal{F}(G)\} < \infty$. Then, by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}
\int_{\mathcal{M}^e(X,G)} \mathcal{F}_*(\theta) dm(\theta) &= \int_{\mathcal{M}^e(X,G)} \lim_{n\to\infty} \frac{1}{|F_n|} \int_X f_{F_n}(x)\,\mathrm{d}\theta(x)\,\mathrm{d}m(\theta) \\
&= \lim_{n\to\infty} \int_{\mathcal{M}^e(X,G)} \frac{1}{|F_n|} \int_X f_{F_n}(x)\,\mathrm{d}\theta(x)\,\mathrm{d}m(\theta) \\
&= \lim_{n\to\infty} \frac{1}{|F_n|} \int_X f_{F_n}(x)\,\mathrm{d}\mu(x) = \mathcal{F}_*(\mu).
\end{aligned} \tag{3.14}$$

Combining Lemma 2.6, (3.14) and (1.2),

$$\begin{aligned}
P(G,\mathcal{F};\mathcal{U}) &= h_\mu(G,\mathcal{U}) + \mathcal{F}_*(\mu) \\
&= \int_{\mathcal{M}^e(X,G)} h_\theta(G,\mathcal{U})\,\mathrm{d}m(\theta) + \int_{\mathcal{M}^e(X,G)} \mathcal{F}_*(\theta)\,\mathrm{d}m(\theta) \\
&= \int_{\mathcal{M}^e(X,G)} \left(h_\theta(G,\mathcal{U}) + \mathcal{F}_*(\theta)\right)\,\mathrm{d}m(\theta).
\end{aligned}$$

Hence there exists $\theta \in \mathcal{M}^e(X,G)$ such that

$$P(G,\mathcal{F};\mathcal{U}) \le h_\theta(G,\mathcal{U}) + \mathcal{F}_*(\theta),$$

which is complete the proof of Theorem 1.1. $\square$

4. An Example

In this section, we main give a nontrivial example of sub-additive potentials that satisfies conditions (C1), (C2) and (C3).

Example 4.1. *Let (X,G) be a G-system and $M : X \to \mathbb{R}^{n\times n}$ a nonnegative continuous matrix function of X, i.e., $M = (M_{i,j})_{n\times n}$, where $M_{i,j}$ are nonnegative continuous of X for all $i,j = 1,2,\cdots,n$.*

Now we define $\mathcal{F} = \{f_E\}_{E\in\mathcal{F}(G)}$ as follows: for each $x \in X$ and $E \in \mathcal{F}(G)$,

$$f_E(x) := \min_{1\le m\le |E|} \min_{(g_i)\in E^m} \log \left\| \prod_{i=1}^m M(g_i x) \right\|,$$

where $\|M(x)\| = \sum_{i,j=1}^n M_{i,j}(x)$. We will show that $\mathcal{F}$ satisfies conditions (C1), (C2) and (C3)

- *$f_{Eg}(x) = f_E(gx)$ for all $x \in X$, $g \in G$ and $E \in \mathcal{F}(G)$;*
- *$f_{E\cup F} \leqslant f_E + f_F$ for all $E, F \in \mathcal{F}(G)$ with $E \cap F = \emptyset$;*
- *Given $E \in \mathcal{F}(G)$ and $g \notin E$. Then there exists $1 \le i \le m \le |E|$ such that*

$$f_{E\cup\{g\}}(x) = \log \|M(g_1x)\cdots M(g_ix)M(gx)M(g_{i+1}x)\cdots M(g_mx)\|.$$

Set $A = M(g_1x)\cdots M(g_ix)$, $B = M(gx)$ *and* $C = M(g_{i+1}x)\cdots M(g_mx)$. *Thus,*

$$f_E(x) - f_{E\cup\{g\}}(x) \leq \log \frac{\|AC\|}{\|ABC\|}.$$

Let

$$K_1 = \frac{\min_{1\leq i,j\leq n}\min_{x\in X} M_{i,j}(x)}{\max_{1\leq i,j\leq n}\max_{x\in X} M_{i,j}(x)}, \quad K_2 = \min_{1\leq i,j\leq n}\min_{x\in X} M_{i,j}(x).$$

Then $K_1, K_2 \in (0, +\infty)$ *and* $M(x) - \frac{K_1}{n}EM(x)$ *is a nonnegative matrix, where* $E = (E_{i,j}), E_{i,j} \equiv 1$. *Hence,*

$$\begin{aligned} \|ABC\| &\geq \left\| A\frac{K_1}{n}EBC \right\| = \frac{K_1}{n}\|A\|\,\|BC\| \\ &\geq \left(\frac{K_1}{n}\right)^2 \|A\|\|BEC\| = \left(\frac{K_1}{n}\right)^2 \|A\|\|B\|\|C\| \\ &\geq \left(\frac{K_1}{n}\right)^2 n^2K_2\|A\|\|C\| = K_1^2K_2\|A\|\|C\|, \end{aligned}$$

which follows that

$$\frac{\|AC\|}{\|ABC\|} \leq \frac{1}{K_1^2K_2} < \infty.$$

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